Lecture 02. Probability

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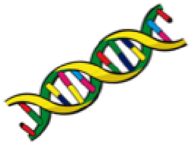
These slides are based on slides from Mahdi Roozbahani

Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes’ Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

Probability

- A sample space S is the set of all possible outcomes of a conceptual or physical, repeatable experiment. (S can be finite or infinite.)
  - E.g., S may be the set of all possible outcomes of a dice roll: S $(1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6)$
  - E.g., S may be the set of all possible nucleotides of a DNA site: S $(A \quad C \quad G \quad T)$



  - E.g., S may be the set of all possible time-space positions of an aircraft on a radar screen.
- An Event A is any subset of S
  - Seeing "1" or "6" in a dice roll; observing a "G" at a site; UA007 in space-time interval

Three Key Ingredients in Probability Theory

A sample space is a collection of all possible **outcomes**

Random variables X represents **outcomes** in sample space

Probability of a random variable to happen $p(x) = p(X = x)$

$p(x) \geq 0$

Continuous variable
Continuous probability distribution
Probability density function
Density or likelihood value
Temperature (real number)
Gaussian Distribution

$$\int_x p(x)dx = 1$$

Discrete variable
Discrete probability distribution
Probability mass function
Probability value
Coin flip (integer)
Bernoulli distribution

$$\sum_{x \epsilon A} p(x) = 1$$

Continuous Probability Functions

- Examples:
  - Uniform Density Function:

$$f_x(x) = \begin{cases} \dfrac{1}{b-a} & for\ a \le x \le b \\ 0 & otherwise \end{cases}$$

  - Exponential Density Function:

$$f_x(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} \qquad for\ x \ge 0$$

$$F_x(x) = 1 - e^{\frac{-x}{\mu}} \qquad for\ x \ge 0$$

  - Gaussian(Normal) Density Function

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Discrete Probability Functions

- Examples:
  - Bernoulli Distribution:
    - $\begin{cases} 1-p & for\ x=0 \\ p & for\ x=1 \end{cases}$

In Bernoulli, just a **single** trial is conducted

  - Binomial Distribution:
    - $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$

k is number of successes

n-k is number of failures

$\binom{n}{k}$ The total number of ways of selection **k** distinct combinations of **n** trials, **irrespective of order**.

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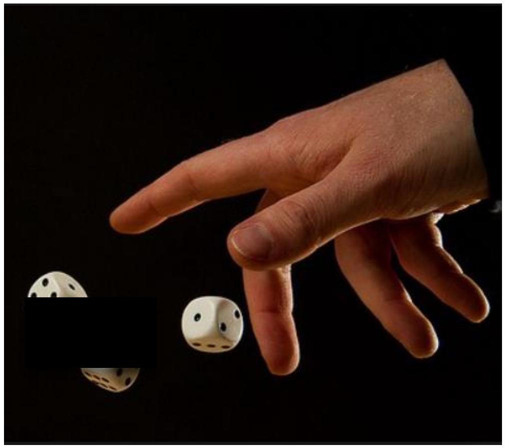
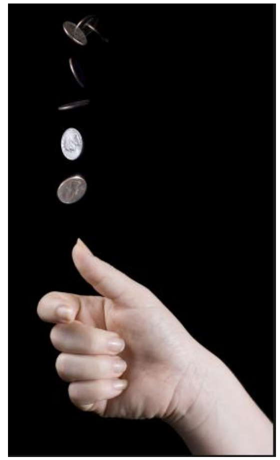
Example



X = Throw a dice



Y = Flip a coin

X and **Y** are random variables

N = total number of trials

$\boldsymbol{n_{ij}}$ = Number of occurrence

X

Y		$x_{i=1} = 1$	$x_{i=2} = 2$	$x_{i=3} = 3$	$x_{i=4} = 4$	$x_{i=5} = 5$	$x_{i=6} = 6$	C_j
	$y_{j=2} = tail$	$n_{ij} = 3$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 5$	$n_{ij} = 1$	$n_{ij} = 5$	20
	$y_{j=1} = head$	$n_{ij} = 2$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 1$	15
	C_i	5	6	6	7	5	6	N=35

X

Y

	$x_{i=1} = 1$	$x_{i=2} = 2$	$x_{i=3} = 3$	$x_{i=4} = 4$	$x_{i=5} = 5$	$x_{i=6} = 6$	C_j
$y_{j=2} = tail$	$n_{ij} = 3$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 5$	$n_{ij} = 1$	$n_{ij} = 5$	20
$y_{j=1} = head$	$n_{ij} = 2$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 1$	15
C_i	5	6	6	7	5	6	N=35

$$p(x = 1, y = tail) =$$
$$p(y = tail|x = 1) =$$
$$p(y = head) =$$

Probability:

$$p(X = x_i) = \frac{c_i}{N}$$

Joint probability:

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Conditional probability:

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

Sum rule

$$p(X = x_i) = \sum_{j=1}^{L} p\left(X = x_i, Y = y_j\right) \Rightarrow \qquad p(X) = \sum_{Y} P(X, Y)$$

Product rule

$$p\left(X = x_i, Y = y_j\right) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i}\frac{c_i}{N} = p\left(Y = y_j \middle| X = x_i\right)p(X = x_i)$$

$$p(X, Y) = p(Y|X)p(X)$$

Conditional Independence

- Examples:

P(Virus | Drink Beer) = P(Virus)
iff Virus is independent of Drink Beer

P(Flu | Virus;DrinkBeer) = P(Flu|Virus)
iff Flu is independent of Drink Beer, given Virus

P(Headache | Flu;Virus;DrinkBeer) =
P(Headache|Flu;DrinkBeer)
iff Headache is independent of Virus, given Flu and Drink Beer

Assume the above independence, we obtain:
P(Headache;Flu;Virus;DrinkBeer)
=P(Headache | Flu;Virus;DrinkBeer) P(Flu | Virus;DrinkBeer)
P(Virus | Drink Beer) P(DrinkBeer)
=P(Headache|Flu;DrinkBeer) P(Flu|Virus) P(Virus) P(DrinkBeer)

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Bayes' Rule

- P(X|Y)= Fraction of the worlds in which X is true given that Y is also true.

- For example:
  - H="Having a headache"
  - F="Coming down with flu"
  - $P(Headche|Flu)$ = fraction of flu-inflicted worlds in which you have a headache. How to calculate?

- Definition:

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} = \frac{\mathrm{P(Y|X)P(X)}}{\mathrm{P(Y)}}$$

Corollary:

$$P(X,Y) = P(Y|X)P(X)$$

This is called Bayes Rule

Bayes' Rule

- $P(Headache|Flu) = \frac{P(Headache,Flu)}{P(Flu)}$

$$= \frac{P(Flu|Headache)P(Headache)}{P(Flu)}$$

Other cases:

- $P(Y|X) = \frac{P(X|Y)P(Y)}{P(X|Y)P(Y)+P(X|\neg Y)P(\neg Y)}$
- $P(Y = y_i|X) = \frac{P(X|Y)P(Y)}{\sum_{i\in S} P(X|Y = y_i)P(Y=y_i)}$
- $P(Y|X, Z) = \frac{P(X|Y, Z)P(Y,Z)}{P(X,Z)} =$

$$\frac{P(X|Y, Z)P(Y,Z)}{P(X|Y, Z)P(Y,Z)+P(X|\neg Y, Z)P(\neg Y,Z)}$$

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Mean and Variance

- Expectation: The mean value, center of mass, first moment:
$$E_X[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx = \mu$$
- N-th moment: $g(x) = x^n$
- N-th central moment: $g(x) = (x - \mu)^n$
- Mean: $E_X[X] = \int_{-\infty}^{\infty} xp_X(x)dx$
  - $E[\alpha X] = \alpha E[X]$
  - $E[\alpha + X] = \alpha + E[X]$
- Variance(Second central moment): $Var(x) = E_X[(X - E_X[X])^2] = E_X[X^2] - E_X[X]^2$
  - $Var(\alpha X) = \alpha^2 Var(X)$
  - $Var(\alpha + X) = Var(X)$

For Joint Distributions

- Expectation and Covariance:
  - $E[X+Y] = E[X] + E[Y]$
  - $cov(X,Y) = E[(X - E_X[X])(Y - E_Y(Y)] = E[XY] - E[X]E[Y]$
  - $Var(X+Y) = Var(X) + 2cov(X,Y) + Var(Y)$

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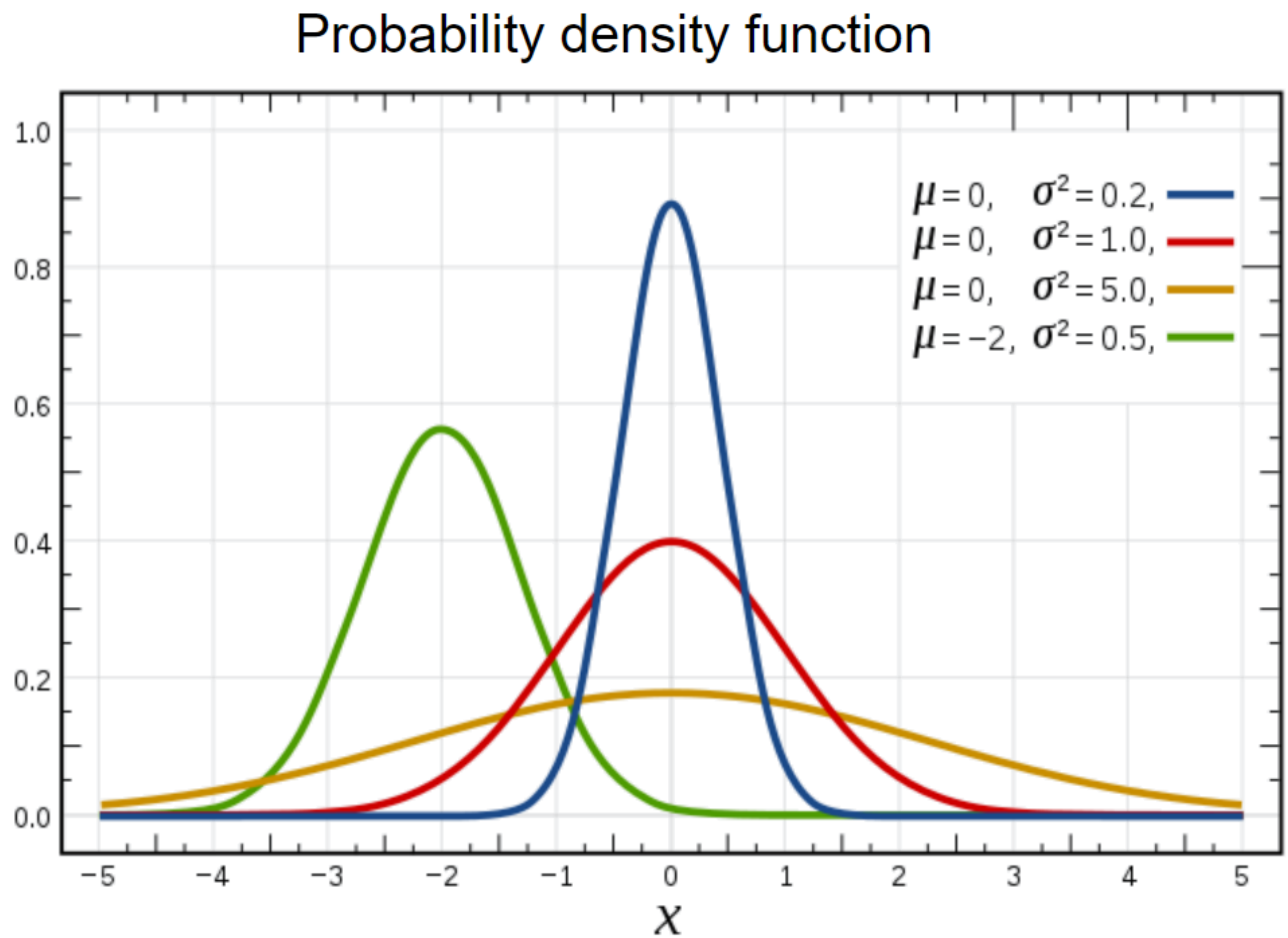
Gaussian Distribution

- Gaussian Distribution: $$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



[Probability versus likelihood]()

Multivariate Gaussian Distribution

$$p(x|\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(x-\mu)^{\top}\Sigma^{-1}(x-\mu)\}$$

- Moment Parameterization $\mu = E(X)$

$$\Sigma = Cov(X) = E[(X-\mu)(X-\mu)^{\top}]$$

- Mahalanobis Distance $\Delta^2 = (x-\mu)^{\top}\Sigma^{-1}(x-\mu)$

- Tons of applications (MoG, FA, PPCA, Kalman filter,...)

Properties of Gaussian Distribution

- The linear transform of a Gaussian r.v. is a Gaussian. Remember that no matter how x is distributed

$$E(AX+b) = AE(X)+b$$
$$Cov(AX+b) = ACov(X)A^\top$$

this means that for Gaussian distributed quantities:

$$X \sim N(\mu,\Sigma) \;\rightarrow\; AX+b \sim N(A\mu+b, A\Sigma A^\top)$$

- The sum of two independent Gaussian r.v. is a Gaussian

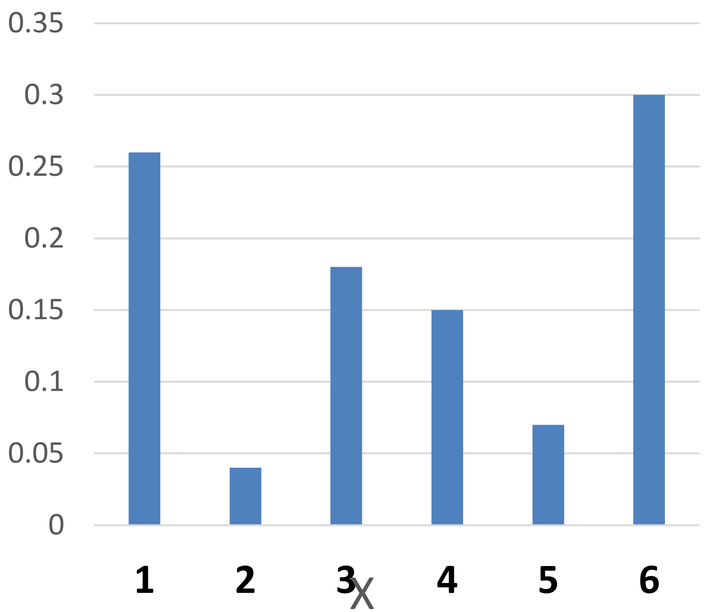
$$\mathrm{Y} = \mathrm{X}_1 + \mathrm{X}_2,\ \mathrm{X}_1 \perp X_2 \rightarrow \mu_y = \mu_1+\mu_2, \Sigma_y = \Sigma_1+\Sigma_2$$

- The multiplication of two Gaussian functions is another Gaussian function (although no longer normalized)

$$N(a,A)N(b,B) \propto N(c,C),$$
$$where\ C = (A^{-1}+B^{-1})^{-1}, c = \mathrm{C}\mathrm{A}^{-1}a + CB^{-1}b$$

Central Limit Theorem

Probability mass function of a **biased** dice



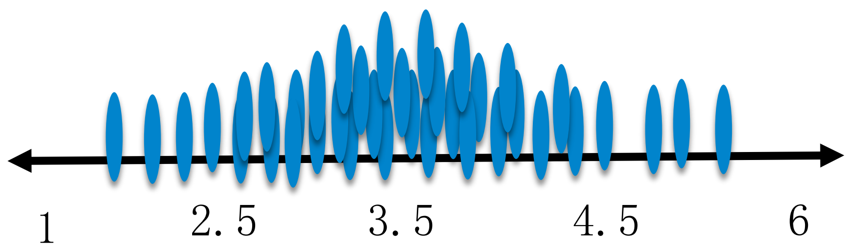
Let's say, I am going to get a sample from this pmf having a size of $\boldsymbol{n = 4}$

$$S_1 = \{1,1,1,6\} \Rightarrow E(S_1) = 2.25$$

$$S_2 = \{1,1,3,6\} \Rightarrow E(S_2) = 2.75$$

$$\vdots$$

$$S_m = \{1,4,6,6\} \Rightarrow E(S_m) = 4.25$$



According to CLT, it will follow a bell curve distribution (normal distribution)

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Maximum Likelihood Estimation

- Probability: inferring probabilistic quantities for data given fixed models (e.g. prob. of events, marginals, conditionals, etc).
- Statistics: inferring a model given fixed data observations (e.g. clustering, classification, regression).

Main assumption:

Independent and identically distributed random variables

i.i.d

Maximum Likelihood Estimation

For Bernoulli (i.e. flip a coin):

Objective function: $f(x_i;p) = p^{x_i}(1-p)^{1-x_i}$ $\quad x_i \in \{0,1\}$ or $\{head, tail\}$

$$L(p) = p(X = x_1, X = x_2, X = x_3, \dots, X = x_n)$$

i.i.d assumption

$$= p(X = x_1)p(X = x_2) \dots p(X = x_n) = f(x_1;p)f(x_2;p) \dots f(x_n;p)$$

$$L(p) = \prod_{i=1}^{n} f(x_i;p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$L(p) = p^{x_1}(1-p)^{1-x_1} \times p^{x_2}(1-p)^{1-x_2} \dots \times p^{x_n}(1-p)^{1-x_n} =$$

$$= p^{\sum x_i}(1-p)^{\sum(1-x_i)}$$

We don’ t like multiplication, let’ s convert it into summation

What’s the trick?

Take the log

$$L(p) = p^{\sum x_i}(1-p)^{\sum(1-x_i)}$$

$$logL(p) = l(p) = \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1-x_i)$$

How to optimize p?

$$\frac{\partial l(p)}{\partial p} = 0$$

$$\frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p} = 0$$

$$p = \frac{1}{n}\sum_{i=1}^{n} x_i$$